3/30/26

Finish §3.4. One of our goals will be to study the following situation:

Th 3.4.7

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a lin transf defined by an $n \times n$ matrix A.

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}\}$ be a basis.

Recall $B[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{B}}$ where $B = \left[[T(\vec{v}_1)]_{\mathcal{B}} \mid \cdots \mid [T(\vec{v}_n)]_{\mathcal{B}} \right]$

Then the $\mathcal{B}$-matrix of T is diagonal, ie

$$B = \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix},$$

iff $T(\vec{v}_1) = c_1 \vec{v}_1$ so $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\vdots$

$T(\vec{v}_n) = c_n \vec{v}_n$ so $[T(\vec{v}_n)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c_n \end{bmatrix}$.

(So T sends each $\vec{v}_i$ to a multiple of itself.)

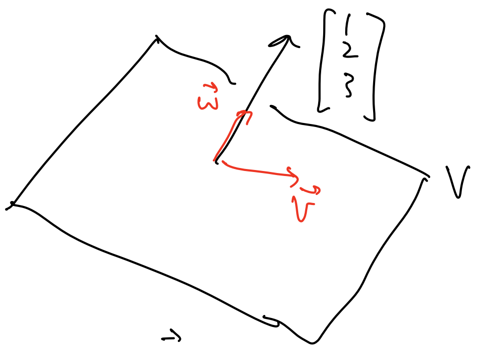
(This is more a def. than a theorem.) Several problems in the HW give you a chance to absorb this idea.

Ex (#39) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the lin transf defined by reflection about the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^3$ so that the $\mathcal{B}$-matrix for T is diagonal.

Sol'n:

We want a basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ so that T sends each to a scalar multiple of itself.

Think geometrically:



let V be the orthogonal plane to $\begin{bmatrix}1\\2\\3\end{bmatrix}$,

namely we want $\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = 0$

so $x + 2y + 3z = 0$

For any $\vec{0} \neq \vec{v} \in V$, $T(\vec{v}) = -\vec{v}$

For any scalar mult. $\vec{w}$ of $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $T(\vec{w}) = \vec{w}$

So pick $\vec{v}_1, \vec{v}_2$ to be any lin. indep elements of V and pick $\vec{v}_3 = \begin{bmatrix}1\\2\\3\end{bmatrix}$.

Specifically, e.g., $\vec{v}_1 = \begin{bmatrix}5\\-1\\-1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}2\\-1\\0\end{bmatrix}$

then $B = \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$

<u>Ex</u> Consider a lin transf $T: \mathbb{R}^2 \to \mathbb{R}^2$.

let $\mathcal{B} = \left\{ \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix} \right\}$

We are told that the matrix for T in terms of $\mathcal{B}$ is

$B = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$. goes from $\mathcal{B}$ coords to $\mathcal{B}$-coords

Find the standard matrix for T.

<u>Sol'n</u>

Use Thm 3.4.4

$S = \begin{bmatrix}-1 & -1\\ 1 & 2\end{bmatrix}$ $\qquad S^{-1} = -\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix} = \begin{bmatrix}-2 & -1\\ 1 & 1\end{bmatrix}$

goes from $\mathcal{B}$ to $\mathcal{S}$ $\qquad\qquad$ goes from $\mathcal{S}$ to $\mathcal{B}$

$$A = S B S^{-1}$$

$$= \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -a-c & -b-d \\ a+2c & b+2d \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2a+2c-b-d & a+c-b-d \\ -2a-4c+b+2d & -a-2c+b+2d \end{bmatrix}$$

We'll skip Ch 4 (Linear Spaces) but the idea is that there are sets that act just like $\mathbb{R}^n$ but don't look like it, and our methods apply equally well to them.

<u>Ex</u> $V = \{$polynomials in x, y of degree $\leq 3\}$

V is closed under $+$ and scalar mult.

Basis $\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$ so it's a vector space of dimension 10.

$T: V \to V$ defined by $T(f) = \dfrac{\partial f}{\partial x}$

is a linear transf. with kernel spanned by $\{1, y, y^2, y^3\}$

and image spanned by $\{1, x, y, x^2, xy, y^2\}$

Move to Ch 5 Orthogonality and Least Squares
(we'll have to pick and choose a bit: 12 classes left, incl. one exam + review)

1st couple of topics:

- what is an orthonormal basis
- why do we want one for a vector space?
- how do we find one (Gram-Schmidt process)

Def

1. Two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are perpendicular (or orthogonal)

   if $\vec{v} \cdot \vec{w} = 0$

2. The length of a vector $\vec{v}$ is $\|\vec{v}\| = \sqrt{(\vec{v} \cdot \vec{v})}$

3. A vector $\vec{u} \in \mathbb{R}^n$ is a unit vector if $\|\vec{u}\| = 1$ (equiv $(\vec{u} \cdot \vec{u}) = 1$)

Recall:

if $\vec{v} \in \mathbb{R}^n$ then $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$ is a unit vector

Def Given a vector $\vec{x} \in \mathbb{R}$ and a subspace $V \subset \mathbb{R}^n$, we say $\vec{v}$ is orthogonal to V if $(\vec{x} \cdot \vec{v}) = 0 \quad \forall \vec{v} \in V$.

Lemma If $\vec{v}_1, \ldots, \vec{v}_m \in V$ form a basis for V and $\vec{x} \in \mathbb{R}^n$, then $\vec{x}$ is orthogonal to V iff $(\vec{x} \cdot \vec{v}_i) = 0$ for $i = 1, \ldots, m$

pf/ will be on HW